Maltsev conditions and directed graphs

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Digraphs, homomorphisms and polymorphisms

Definition

A **digraph** is a pair $\mathbb{G} = (G; \to)$, where G is the set of **vertices** and $\to \subseteq G^2$ is the set of **edges**.

Definition

A **homomorphism** from $\mathbb{G}$ to $\mathbb{H}$ is a map $f : G \to H$ that preserves edges:

$$a \to b \text{ in } \mathbb{G} \quad \Longrightarrow \quad f(a) \to f(b) \text{ in } \mathbb{H}.$$

$\mathrm{Hom}(\mathbb{G}, \mathbb{H})$ is the set of all homomorphisms from $\mathbb{G}$ to $\mathbb{H}$.

Definition

The clone of **polymorphisms** of $\mathbb{G}$ is $\mathrm{Hom}(\mathbb{G}) = \bigcup_{n=1}^{\infty} \mathrm{Hom}(\mathbb{G}^n, \mathbb{G})$.

Gumm polymorphisms of digraphs

Theorem (Larose, Zádori; 1997)

If a finite poset (reflexive, transitive, antisymmetric digraph) has Gumm polymorphisms

$$\begin{aligned} x &\approx d_0(x,y,z), \\ d_i(x,y,x) &\approx x \text{ for all } i, \\ d_i(x,y,y) &\approx d_{i+1}(x,y,y) \text{ for even } i, \\ d_i(x,x,y) &\approx d_{i+1}(x,x,y) \text{ for odd } i, \\ d_n(x,y,y) &\approx p(x,y,y), \text{ and} \\ p(x,x,y) &\approx y, \end{aligned}$$

then it has a near-unanimity polymorphism

$$n(y,x,\dots,x) \approx \cdots \approx n(x,\dots,x,y) \approx x.$$

Gumm polymorphisms of digraphs

Theorem (Larose, Loten, Zádori; 2005)

If a finite reflexive and symmetric digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism.

Theorem (M, Zádori; 2012)

If a finite reflexive digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism and totally symmetric polymorphisms

$$\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\} \implies t(x_1, \ldots, x_n) \approx t(y_1, \ldots, y_n)$$

for all arities.

Theorem

If a finite symmetric digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism.

How far can we push this?

Theorem (Kazda; 2011)

If a finite digraph has a Maltsev polymorphism

$$p(x,x,y) \approx p(y,x,x) \approx y,$$

then it admits a majority polymorphism

$$m(y,x,x) \approx m(x,y,x) \approx m(x,x,y) \approx x.$$

Theorem (Bulín, Delić, Jackson, Niven; 2013)

For every finite relational structure $\mathbb{A}$ there exists a finite directed graph $\mathbb{G}$, such that almost all Maltsev conditions (Taylor term, Willard terms, Hobby-McKenzie terms, Gumm terms, edge term, Jonsson terms, near-unanimity term, but not Maltsev term) hold equivalently by $\mathbb{A}$ and $\mathbb{G}$.

Connectivity

Definition

$\mathbb{G}$ is **strongly connected** if for any $a, b \in G$ there exists a **directed path** $a = a_0 \to a_1 \to \cdots \to a_n = b$ of length $n \geq 0$. $\mathbb{G}$ is **connected** if for any $a, b \in G$ there exists an **oriented path** $a = a_0 \to a_1 \leftarrow \cdots \to a_n = b$ of length $n \geq 0$ where the arrows can point either way. The digraph $\mathbb{G}$ is **smooth**, if its edge relation is subdirect (no sources and sinks).

Definition

The [strong, smooth] components of $\mathbb{G}$ are the maximal [strong, smooth] induced subgraphs of $\mathbb{G}$.

Definition

The **algebraic length** of a directed path is the number of forward edges minus the number of backward edges. The algebraic length of $\mathbb{G}$ is the smallest positive algebraic length of oriented cycles (closed paths) of $\mathbb{G}$.

Algebraic length 1

Proposition

If $\mathbb{G}, \mathbb{H}$ are connected, smooth and $\mathbb{G}$ has algebraic length 1, then $\mathbb{G} \times \mathbb{H}$ is connected and smooth.

Proof.

- for any $a \in G$ there exists an oriented cycle of algebraic length 1 going through a
- for any $a \to b$ in $\mathbb{G}$ there exists an oriented path from a to b of algebraic length 0
- for any $x \in H$, (a, x) and (b, x) are connected in $\mathbb{G} \times \mathbb{H}$ □

Proposition

If $\mathbb{G}$ is smooth, algebraic length 1 and (strongly) connected, then $\mathbb{G}^n$ is smooth, algebraic length 1 and (strongly) connected for all $n \geq 1$.

Cores

Definition

Write $\mathbb{G} \to \mathbb{H}$ iff there exists a homomorphism from $\mathbb{G}$ to $\mathbb{H}$.

Proposition

$\to$ is a quasi-order on the set of finite digraphs. If $\mathbb{G}$ is a minimal member of the $\leftrightarrow$ class of $\mathbb{H}$, then

- *every endomorphism of $\mathbb{G}$ is an automorphism,*
- *$\mathbb{G}$ is uniquely determined up to isomorphism, and*
- *$\mathbb{G}$ is isomorphic to a induced substructure of $\mathbb{H}$.*

Definition

$\mathbb{G}$ is a **core** if it has no proper endomorphism. The **core of** $\mathbb{H}$ is the uniquely determined core structure in the $\leftrightarrow$ class of $\mathbb{H}$.

The loop lemma

Theorem (Barto, Kozik, Niven; 2008)

If $\mathbb{G}$ is smooth, algebraic length 1, and has a Taylor polymorphism, or equivalently a weak near-unanimity polymorphism

$$w(x,\dots,x) \approx x \quad \text{and} \quad w(y,x,\dots,x) \approx \dots \approx w(x,\dots,x,y),$$

then $\mathbb{G}$ has a loop.

Corollary

The core of a smooth digraph with a Taylor polymorphism is a disjoint union of cycles.

Problem

Let $\mathbb{G}$ be a smooth, connected, algebraic length 1 digraph that has Gumm polymorphisms. Does $\mathbb{G}$ need to have a near-unanimity polymorphism?

Maltsev digraphs again

Theorem

If $\mathbb{G} = (G; E)$ is smooth, connected, algebraic length 1, and has Maltsev polymorphism, then it has join and meet polymorphisms.

Proof.

- $\alpha = E \circ E^{-1}$ and $\beta = E^{-1} \circ E$ are equivalence relations (congruences)
- case 1: $(a, b) \in \alpha \wedge \beta$ and $a \neq b$
  - $r \colon \mathbb{G} \to \mathbb{G} \setminus \{b\}$, $r(x) = x$ for $x \neq b$ and $r(b) = a$ is a retraction
  - by induction we have join and meet polymorphisms on $r(\mathbb{G})$
  - we can extend them to $\mathbb{G}$ by splitting $\{a, b\}$ into $a < b$
- case 2: $\alpha \wedge \beta = 0$
  - by induction the digraph $\mathbb{G}/\alpha = (G/\alpha; E/\alpha)$ has join and meet
  - the digraph $\mathbb{E}/\alpha = (E/\alpha; \pi_2 \circ \pi_1^{-1})$ has join and meet
  - the digraphs $\mathbb{E}/\alpha$ and $\mathbb{G}$ are isomorphic via the map $\varphi \colon E/\alpha \to G$, $\varphi(x/\alpha, y/\alpha) = x/\beta \cap y/\alpha$

□

Exponentiation

Definition

Let $\mathbb{H}^{\mathbb{G}}$ be the digraph on the set H^G with edge relation $f \to g$ iff

$$a \to b \text{ in } \mathbb{G} \implies f(a) \to g(b) \text{ in } \mathbb{H}.$$

Proposition

- $\mathrm{Hom}(\mathbb{G}, \mathbb{H}) = \{f \in \mathbb{H}^{\mathbb{G}} : f \to f\}$
- $\mathbb{G}^n = \mathbb{G}^{\mathbb{L}_n}$ *where* $\mathbb{L}_n = (\{1, \dots, n\}; =)$
- $\left(\mathbb{H}^{\mathbb{G}}\right)^{\mathbb{F}} = \mathbb{H}^{\mathbb{G} \times \mathbb{F}}$
- $\mathbb{H}^{\mathbb{F}} \times \mathbb{G}^{\mathbb{F}} = (\mathbb{H} \times \mathbb{G})^{\mathbb{F}}$
- *the composition map* $\circ : \mathbb{H}^{\mathbb{G}} \times \mathbb{G}^{\mathbb{F}} \to \mathbb{H}^{\mathbb{F}}$ *is a homomorphism*
- *If* $f \to g$ *in* $\mathbb{H}^{\mathbb{G}^n}$ *and* $f_1 \to g_1, \dots, f_n \to g_n$ *in* $\mathbb{G}^{\mathbb{F}}$, *then*

$$f(f_1, \dots, f_n) \to g(g_1, \dots g_n) \text{ in } \mathbb{H}^{\mathbb{F}}$$

Exponentiation in finite duality

- set of finite relational structures modulo $\leftrightarrow$ is a partially ordered set
- isomorphic to the set of core isomorphism types
- minimal [maximal] element: 1-element structure, with empty [full] relations
- join: disjoint union, meet: direct product,
- satisfies distributive laws, join irreducible = connected
- Heyting algebra (relatively pseudocomplemented)
- $\mathbb{F} \wedge \mathbb{G} \leq \mathbb{H} \iff \mathbb{H}^{\mathbb{F}\times\mathbb{G}} = \left(\mathbb{H}^{\mathbb{G}}\right)^{\mathbb{F}}$ has a loop $\iff \mathbb{F} \leq \mathbb{H}^{\mathbb{G}}$
- if $\mathbb{G}$ is join irreducible with lower cover $\mathbb{H}$, then $(\mathbb{G}, \mathbb{H}^{\mathbb{G}})$ is a dual pair

Theorem (Nešetřil, Tardif, 2010)

Let $\mathbb{G}$ be a finite connected core structure. Then $\mathbb{G}$ has a dual pair $\mathbb{H}$, i.e. $\{\,\mathbb{F} \mid \mathbb{F} \to \mathbb{G}\,\} = \{\,\mathbb{F} \mid \mathbb{H} \not\to \mathbb{F}\,\}$, if and only if $\mathbb{G}$ is a tree.

$\mathrm{End}(\mathbb{G})$, $\mathrm{Sym}(\mathbb{G})$ AND $\mathrm{Aut}(\mathbb{G})$

Definition

$\mathrm{End}(\mathbb{G})$ and $\mathrm{Sym}(\mathbb{G})$ are the induced subgraphs of $\mathbb{G}^{\mathbb{G}}$ on $\mathrm{Hom}(\mathbb{G}, \mathbb{G})$ and the set of permutations, respectively. $\mathrm{Aut}(\mathbb{G}) = \mathrm{End}(\mathbb{G}) \cap \mathrm{Sym}(\mathbb{G})$.

Proposition

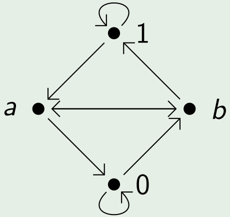
The components of $\mathrm{Sym}(\mathbb{G})$ $[\mathrm{End}(\mathbb{G})]$ *that contain an automorphism are isomorphic to the component of the identity.*

Theorem (Gyenizse; 2013)

$\mathrm{Aut}(\mathbb{G})$ *is a disjoint union of complete digraphs. Moreover, the number of elements in each component is the same and is a product of factorials.*

Example

The following digraph $\mathbb{G}$ has Maltsev, join and meet semilattice polymorphisms.



It has only four endomorphisms: id, 0, 1 and inversion, they are all isolated. However, id is connected to 0 in $\mathbb{G}^{\mathbb{G}}$:

$$\mathrm{id} = x \wedge 1 \to x \wedge a \to x \wedge 0 = 0.$$

Connectivity in $\mathbb{G}^{\mathbb{G}}$

Theorem (M, Zádori; 2012)

If $\mathbb{G}$ is a connected reflexive digraph with Hobby-McKenzie polymorphisms, then $\mathrm{End}(\mathbb{G})$ is connected.

Theorem (Gyenizse; 2013)

Suppose, that $|\mathbb{G}| \geq 6$. Then $\mathbb{G}^{\mathbb{G}}$ is connected if and only if

- *$\mathbb{G}$ is empty,*
- *there exists $a \in G$ such that $a \to x$ for all $x \in G$, or*
- *there exists $a \in G$ such that $x \to a$ for all $x \in G$.*

Theorem (Gyenizse; 2013)

If $|\mathbb{G}| \geq 6$ and $\mathrm{Sym}(\mathbb{G})$ is connected, then $\mathbb{G}^{\mathbb{G}}$ must be also connected.

The component of the identity

Definition

A map $f \in \mathbb{G}^{\mathbb{G}}$ is **idempotent**, if $f^2 = f$, it is a **retraction**, if $f \to f$ and $f^2 = f$, and it is **proper**, if $f \neq \mathrm{id}$.

Lemma (M, Zádori; 2012)

If $\mathbb{G}$ is reflexive or symmetric and the component of the identity in $\mathrm{End}(\mathbb{G})$ *contains something other than* id, *then it contains a proper retraction.*

Theorem

If the smooth component of id *in $\mathbb{G}^{\mathbb{G}}$ (or in any submonoid) contains a non-permutation, then it contains a proper retraction.*

Proposition

If $\mathbb{G}$ is smooth and the component of id *contains a constant map, then the smooth part of $\mathbb{G}^{\mathbb{G}}$ is connected (and $\mathbb{G}$ is connected and contains a loop).*

The component of the identity

Example

The digraph $\mathbb{G} = (\{0,1,2\}; \neq)$ with 6 edges is connected, smooth, has algebraic length 1, and the identity in $\mathbb{G}^{\mathbb{G}}$ is isolated.

Example

Let $\mathbb{H} = (H; E)$ be the example with Maltsev, join and meet morphisms:

$$H = \{0,1\}^2 \quad \text{and} \quad E = \{\,(x,y,u,v) \in H^2 \mid y = u\,\}.$$

Then the component of the identity for $\mathbb{G} \times \mathbb{H}$ is non-trivial (isomorphic to $\mathbb{H}$), but it does not contain a non-permutation.

Problem

Find a nontrivial smooth, connected, algebraic length 1 digraph with Taylor polymorphism for which id is isolated in $\mathbb{G}^{\mathbb{G}}$.

Unary polynomials of $\mathbb{G}$

Definition

$\mathrm{Pol}_1(\mathbb{G})$ is the induced subgraph of $\mathbb{G}^{\mathbb{G}}$ on the set of **unary polynomials** of the algebra $\mathbf{G} = (G; \mathrm{Hom}(\mathbb{G}))$.

Proposition

- $\mathrm{Pol}_1(\mathbb{G}) \leq \mathbf{G}^G$ *is generated by the identity and the constant maps*
- $\mathbb{G}$ *is an induced subgraph of* $\mathrm{Pol}_1(\mathbb{G})$ *on the set of constant maps*
- $\mathrm{Pol}_1(\mathbb{G})$ *is smooth if and only if* $\mathbb{G}$ *is smooth*
- *If* $\mathbb{G}$ *is smooth, connected and algebraic length* 1, *then every component of* $\mathrm{Pol}_1(\mathbb{G})$ *has algebraic length* 1

Proof.

For a polynomial $p = t(x, a_1, \dots, a_n)$ we can find an oriented cycle in $\mathbb{G}^n$ of algebraic length 1 going through $(a_1, \dots, a_n)$. Then the polymorphism $t \in \mathrm{Hom}(\mathbb{G}^{n+1}, \mathbb{G}) = \mathrm{Hom}(\mathbb{G}^n, \mathbb{G}^{\mathbb{G}})$ maps this cycle to a cycle in $\mathrm{Pol}_1(\mathbb{G})$.

Twin polynomials

Proposition

If $\mathbb{G}$ is smooth, connected and algebraic length 1, then the connectedness relation on $\mathrm{Pol}_1(\mathbb{G})$ is a congruence.

Definition

Let $\mathbf{A}$ be an algebra. Two unary polynomials $p, q \in \mathrm{Pol}_1(\mathbf{A})$ are **twins**, if there exists a term t of arity $n+1$ and constants $\bar{a}, \bar{b} \in A^n$ such that

$$p = t(x, \bar{a}) \quad \text{and} \quad q = t(x, \bar{b}).$$

The transitive closure of twin polynomials is the **twin congruence** τ of the algebra $\mathrm{Pol}_1(\mathbf{A})$.

Corollary

If $\mathbb{G}$ is smooth, connected and algebraic length 1, then the twin congruence blocks are connected.

Connectivity for SD($\vee$) digraphs

Theorem

*If $\mathbb{G}$ is smooth, connected and algebraic length 1, and the corresponding algebra $\mathbf{G} = (G; \mathrm{Hom}(\mathbb{G}))$ generates a congruence join semi-distributive variety (omits types **1**, **2** and **5**), then $\mathrm{Pol}_1(\mathbb{G})$ is connected.*

Proof.

- for $a \in G$ let η_a be the projection kernel of $\mathrm{Pol}_1(\mathbb{G})$ onto its a-th coordinate
- for any $a \in G$ and $p, q \in \mathrm{Pol}_1(\mathbb{G})$ we have $p\ \eta_a\ p(a)\ \tau\ q(a)\ \eta_a\ q$, so $\tau \vee \eta_a = 1$
- use join semi-distributivity

$$\tau \vee \alpha = \tau \vee \beta \implies \tau \vee \alpha = \tau \vee (\alpha \wedge \beta)$$

to derive $\tau \vee (\bigwedge_a \eta_a) = 1$, that is $\tau = 1$. □

STRUCTURE OF $\mathrm{Pol}_1(\mathbb{G})$

Problem

Let $\mathbb{G}$ be smooth, connected, algebraic length 1 and with Taylor polymorphism. Describe the structure of $\mathrm{Pol}_1(\mathbb{G})$ modulo connectivity.

- We can assume that all twins of the identity are permutations
- The component of the identity has compatible join and meet
- From the loop lemma, every component that contains an idempotent has a loop, that is a proper retraction.

Theorem

If $\mathbb{G}$ is smooth, connected, algebraic length 1 and $\mathbf{G} = (G; \mathrm{Hom}(\mathbb{G}))$ generates a congruence modular variety, then $\mathrm{Pol}_1(\mathbb{G})$ is connected.

Conjecture

If $\mathbb{G}$ is smooth, connected, algebraic length 1 and has Hobby-McKenzie polymorphisms for omitting types **1** and **5**, then $\mathrm{Pol}_1(\mathbb{G})$ is connected.

Connectivity in $\mathrm{Pol}_2^{\mathrm{id}}(\mathbb{G})$

Theorem (M, Zádori; 2012)

If $\mathbb{G}$ is reflexive, connected and has Gumm polymorphisms, then π_1 and π_2 are connected in the graph $\mathrm{Hom}^{\mathrm{id}}(\mathbb{G}^2, \mathbb{G})$ of idempotent binary morphisms.

Theorem

If $\mathbb{G}$ is a smooth, connected, algebraic length 1 digraph which has Gumm polymorphisms, then the digraph $\mathrm{Pol}_2^{\mathrm{id}}(\mathbb{G})$ on the set of idempotent binary polynomials of $\mathbb{G}$ is connected (π_1 and π_2 are connected).

Proof.

Take a path $\mathrm{id} = f_0 \sim f_1 \sim \cdots \sim f_k = c$ in $\mathrm{Pol}_1(\mathbb{G})$ for some constant c.

$$d_i(x,x,y) = d_i(x, f_0(x), y) \sim d_i(x, f_1(x), y) \sim \cdots \sim d_i(x, f_k(x), y)$$
$$= d_i(x,c,y) = d_i(x, f_k(y), y) \sim \cdots \sim d_i(x, f_0(y), y) = d_i(x,y,y), \text{ and}$$
$$p(x,y,y) = p(f_0(x), f_0(y), y) \sim p(f_1(x), f_1(y), y) \sim \cdots \sim p(c,c,y) = y.$$

Idempotent subalgebras

Definition

An induced subgraph $\mathbb{K}$ of $\mathbb{G}^{\mathbb{H}}$ is an **idempotent** $\mathbb{G}$-**subalgebra**, if K is closed under the idempotent polynomials of $\mathbb{G}$.

(Connection to CD absoption...)

Proposition

If $\mathrm{Pol}_2^{\mathrm{id}}(\mathbb{G})$ *is connected then every smooth idempotent subalgebra of* $\mathbb{G}^{\mathbb{H}}$ *is connected.*

Musings

- Can we do something similar for arbitrary relational structures? What are the right notions of smoothness and algebraic length 1?
- Combinatorial vs. algebraic arguments
- We do not even have a complete connectivity description for reflexive or symmetric digraphs...
- How can we adapt the absoption work of Barto and Kozik from the context of $\mathbf{R} \leq \mathbf{A} \times \mathbf{B}$?
- Describe absorption in tame congruence theoretic terms.
- Relations to CSP: consistent set of maps, preserving solutions, maximal absorbing subuniverses vs. maximal idempotents, etc.

Thank You!